Penetration of Arbitrary Double Potential Barriers with Probability Unity: Implications for Testing the Existence of a Minimum Length

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Quantum tunneling across double potential barriers is studied. With the assumption that the real space is a continuum, it is rigorously proved that large barriers of arbitrary shapes can be penetrated by low-energy particles with a probability of unity, i.e., realization of resonant tunneling (RT), by simply tuning the inter-barrier spacing. The results are demonstrated by tunneling of electrons and protons, in which resonant and sequential tunneling are distinguished. The critical dependence of tunneling probabilities on the barrier positions not only demonstrates the crucial role of phase factors, but also points to the possibility of ultrahigh accuracy measurements near resonance. By contrast, the existence of a nonzero minimum length puts upper bounds on the barrier size and particle mass, beyond which effective RT ceases. A scheme is suggested for dealing with the practical difficulties arising from the delocalization of particle position due to the uncertainty principle. This work opens a possible avenue for experimental tests of the existence of a minimum length based on atomic systems.

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I. INTRODUCTION

The scenario of a minimum length (L_{min}) plays an essential role in the quantum theory of gravity [1-11]. Breakdown of the Lorentz invariance may happen when the real space approaches such a minimum length scale [12-20], which is generally taken to be the Planck length ($l_P = \sqrt{\frac{\hbar G}{c^3}} \sim 1.6 \times 10^{-35}\ m$, $\hbar$ is the reduced Planck's constant, G is the gravitational constant and c is the speed of light) [7]. Despite numerous efforts [1-20], the question remains open regarding the existence of such a minimum length [7, 21-23]. Since such a length scale is well below the lower bound of spatial resolution achieved by state-of-the-art instruments such as LIGO ($\sim 10^{-19}\ m$) [24, 25], it is a great challenge for experimental verification. Here, we show the possibility of tackling this problem by investigations of quantum tunneling across double potential barriers.

Quantum tunneling [26] is a classically forbidden phenomenon in which a particle passes through a potential barrier higher than the energy it possesses. In the early years of quantum mechanics, the theories based on quantum tunneling explain some puzzles of experimental observations like the thermionic and field-induced emission of electrons from metal surfaces [26], and the alpha decay of heavy nuclei [26]. In the ensuing decades, researches on the quantum tunneling of electrons in condensed matter have led to fruitful discoveries [27-33], and enabled important inventions such as the scanning tunneling microscope (STM) [34] and tunneling diodes [35-37]. Since the pioneering works by Tsu, Esaki and Chang [31-33], double barriers have received a lot of attention while studying electron transport in heterostructures [37, 38]. Resonant tunneling (RT) typically takes place in double-barrier systems, in which the incident electrons may pass through the barriers without being reflected, i.e., with a transmission probability of 100%. Such a behavior is due to the coherent interference of electron waves which cancel the reflected waves and enhance the transmitted ones, analogous to the resonant transmission through a Fabry-Perot etalon in optics. Typical inter-barrier spacing of the devices based on RT is several tens of angstroms (Å), matching the de Broglie wavelengths of electrons. In recent decades, the phenomena

of RT in mesoscopic and nanoscale structures continue to attract interests of research [39-43].

Historically, RT of electrons was considered to gain experimental evidence from the negative differential resistance (NDR) found in the current-voltage (*I*-*V*) curves [32, 35, 37]. Later, alternative mechanism was suggested for NDR, namely, sequential tunneling in which the phase memory of electron wave functions is lost due to inelastic scattering [37, 44-51]. It was argued that resonant (coherent) tunneling is a prerequisite for sequential tunneling [51]. The effects of external electric field, inelastic scattering, and the repulsive interactions between electrons on RT were also studied [38, 52, 53]. In spite of these efforts, consensus on the underlying physics is yet to be reached. There are still large discrepancies between theory and the measured *I*-*V* curves (e.g., peak-to-valley ratio). The gap originates partly from the fact that, in calculations related to experiments the simplest rectangular barriers (or their variants) are adopted, which usually differ significantly from the true barriers felt by electrons.

To resolve the puzzles, *exact theoretical description of the conditions for RT* across double barriers is highly desired. For the simplest rectangular double barriers, exact mathematical relation of energy and geometric conditions has been established [38, 54, 55]. For the more general and realistic situation where double barriers are of arbitrary shapes, aside from the semi-classical approach [38], full quantum level description of the RT conditions is still lacked. It is generally accepted that RT takes place when the energy of incident particle matches the energy levels of the quasi-bound states within the potential well in-between the two barriers [37, 38, 44, 49, 51, 52]. In principle, this applies to electrons as well as the massive particles like protons, atoms and molecules. Recent simulations have shown the RT of H and He atoms across small double barriers, with the barrier height E_b ~ 0.2 eV [56, 57] and ~ 0.02 eV [58], respectively. However, when a particle tunnels across arbitrarily-shaped double-barriers, it is unclear how the level-match condition can be reached, and rigorous theoretical descriptions remain elusive.

In this paper, we revisit this topic in double-barrier systems consisting of equal barriers of arbitrary geometries. With the assumption of a continuously varied

inter-barrier spacing (equivalently, $L_{min} = 0$), it is rigorously proved that quantum tunneling through the double-barrier system with a probability of unity can always happen (i.e., RT) when the inter-barrier spacing is appropriately chosen. Exact mathematical relation for RT is established. At the presence of a nonzero L_{min} ($L_{min} > 0$), the inter-barrier spacing varies discontinuously, which sets upper bounds for the barrier heights and particle mass, above which no RT may happen. The results are demonstrated by the tunneling of electrons, protons, and some typical bosons. Practically possible scheme is therefore provided for experimental tests of the existence of a nonzero minimum length.

The rest of this paper is organized as follows. Section II presents the analytic and numerical results on RT, with examples of typical particles like electrons, protons and some bosons. The connection between RT and the continuity of real space is revealed. The constraints set by the existence of a minimum length, the practical obstacles due to the uncertainty principle and plausible solutions are presented. We conclude in Sec. III with discussions on the impacts and future opportunities inspired by this work.

II. RESULTS AND DISCUSSIONS

We begin in Part A of this section by performing general analysis on the transmission properties of quantum particles across double barriers of arbitrary shapes, and prove a theorem which establishes the mathematical condition for resonant tunneling (RT). Part B provides analyses on two typical models — rectangular and parabolic double barriers. The quantum tunneling of electrons and protons are studied and compared, with emphasis on the differences between resonant and sequential tunneling. Based on the results of Parts A and B, we show in Part C the upper bounds of barrier heights for RT set by the Planck length. In Part D, the fundamental limits put by the uncertainty principle are studied and possible solution to position delocalization of the incident particles is suggested.

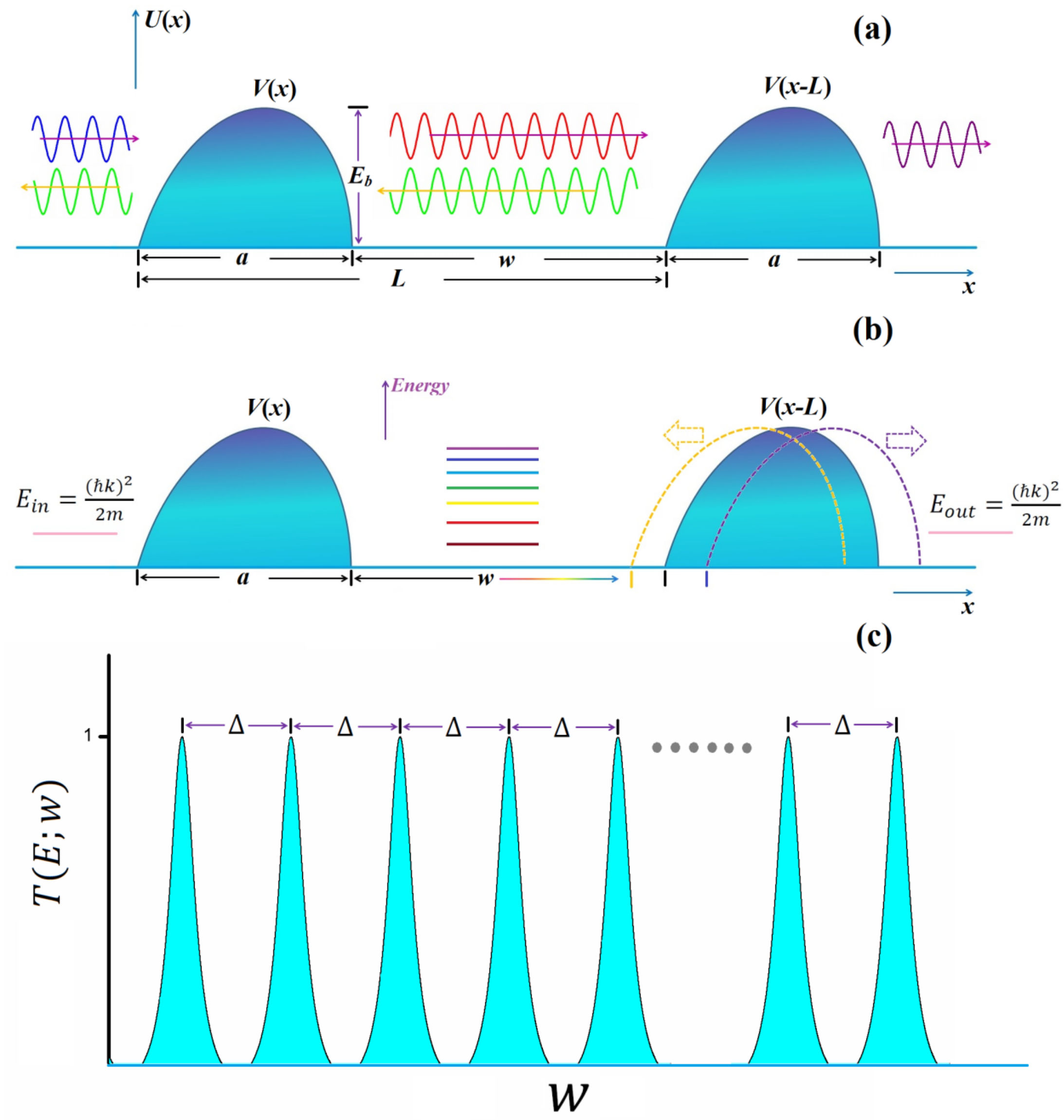

FIG. 1. Schematics of resonant tunneling (RT) across double barriers. **(a)** Quantum interference of the incident and reflected matter waves; **(b)** Modulation of the energy levels of the quasi-bound states in-between the two barriers, by varying the inter-barrier spacing *w*; **(c)** RT spectrum as a function of *w* with a period of $\Delta\,(=\frac{\pi}{k})$.

A. GENERAL ANALYSES ON ARBITARY DOUBLE BARRIERS

Generally, double-barriers consist of two identical or different single barriers, which are respectively referred to as homo-structured and hetero-structured hereafter. The double-barrier considered here is homo-structured in one-dimensional space, as shown in Fig. 1(a), with a barrier height E_b and barrier width *a* for each. Our analyses

are based on the transfer matrix method, a powerful technique for studying the transmission properties in finite systems [31, 59-62]. For the propagation of a quantum particle across a single barrier $V(x)$, the transmitted and reflected amplitudes (A_L, B_L; A_R, B_R) of the wave functions (ψ_L, ψ_R) may be related by a transfer matrix (denoted by M) as follows [38, 56, 59-62]:

$$\begin{pmatrix} A_R \\ B_R \end{pmatrix} = M \begin{pmatrix} A_L \\ B_L \end{pmatrix} \equiv \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} A_L \\ B_L \end{pmatrix}. \tag{1}$$

The incoming wave function (with incident energy E) is $\psi_L = A_L e^{ikx} + B_L e^{-ikx}$, and the outgoing wave function is $\psi_R = A_R e^{ikx} + B_R e^{-ikx}$, where $k = \sqrt{2mE/\hbar^2}$, m is the particle mass. The determinant $|M| = 1$, for systems where time-reversal symmetry preserves, and the transmission coefficient is given by [56] $T = \frac{1}{|m_{11}|^2} = \frac{1}{|m_{22}|^2}$. In general, the matrix elements m_{ij} (i, j = 1, 2) are complex numbers and obey the conjugate relations [56, 59-62] of $m_{11}^* = m_{22}$, and $m_{12}^* = m_{21}$. For a homo-structured double-barrier with an inter-barrier spacing w, the following theorem holds:

Theorem. – For any $E < E_b$, the transmission coefficient (tunneling probability) across a homo-structured double-barrier $T_{DB}(E; w) = 1$ at $w = w_n = \frac{n\pi}{k} - \frac{\pi+\theta+2ka}{2k}$, where $\theta = \arg(m_{11}{}^2)$, n (referred to as resonance number) belongs to integers.

Proof. – The updated transfer matrix for a single barrier $V(x)$ translated by a distance $L = a + w$, $V(x\text{-}L)$, is given by [57, 63]

$$M(L) = \begin{pmatrix} m_{11} & m_{12} e^{-i2kL} \\ m_{21} e^{i2kL} & m_{22} \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} e^{-i2k(a+w)} \\ m_{21} e^{i2k(a+w)} & m_{22} \end{pmatrix}. \tag{2}$$

The transfer matrix for the double-barrier ($U(x) = V(x) + V(x\text{-}L)$) is therefore [56]

$$M_{DB} = M(L) * M = \begin{pmatrix} m_{11} & m_{12} e^{-i2k(a+w)} \\ m_{21} e^{i2k(a+w)} & m_{22} \end{pmatrix} \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}. \tag{3}$$

The diagonal matrix element describing the transmission properties, $(M_{DB})_{11}$, is explicitly calculated to be $(M_{DB})_{11} = m_{11}{}^2 + m_{12} m_{21} e^{-i2k(a+w)}$. Let $Z = m_{11}{}^2 \equiv |Z| e^{i\theta}$, $\phi = 2k(a + w)$, $m_{12} m_{21} = |m_{12}|^2 = R$, with i the imaginary unit, and the angle $\theta = \arg(Z)$; the determinant, $|M| = 1 = |m_{11}|^2 - |m_{12}|^2 = |Z| - R$ gives

that $|Z| = 1 + R$, then $(M_{DB})_{11} = (1 + R)e^{i\theta} + Re^{-i\phi} = e^{i\theta} + Re^{i\theta}(e^{-i(\phi+\theta)} + 1)$. When $e^{-i(\phi+\theta)} = -1$, i.e., $\phi + \theta = (2n - 1)\pi$, with n being integers, one has $(M_{DB})_{11} = e^{i\theta}$. It follows that the transmission coefficient $T_{DB}(E; w) = \frac{1}{|(M_{DB})_{11}|^2} = 1$, which corresponds to RT. Using the condition $\phi + \theta = (2n - 1)\pi$, one has $2k(a + w) + \theta = (2n - 1)\pi$, and consequently, $w = \frac{n\pi}{k} - \frac{\pi+\theta+2ka}{2k} \equiv w_n$. This completes the proof of the theorem.

It should be stressed here that the proof inherently includes the precondition that the inter-barrier spacing w varies continuously (L_{min} = 0) such that the angle ϕ can have any desired values to satisfy the RT condition. The theorem points to the possibility of *penetration of arbitrarily large (but finite) potential barriers by low-energy particles with a probability of unity*. For a quantum particle with incident energy E, it can completely tunnel across a homo-structured double barrier of height E_b when the inter-barrier spacing equals w_n described above, even in the case $E << E_b$. In addition, one sees that the barrier-barrier separations (w_n) for RT are solely determined by the parameters (θ, a) describing the transmission of single barriers. Physically, the onset of RT is due to the presence of quasi-bound states in-between the two barriers whose energy levels match that of the incident particles [37, 38, 44, 49, 51, 52]. A direct consequence is that, any quasi-bound energy levels ($E \leq E_b$) can be realized within the potential well set by the two barriers via simply tuning the inter-barrier spacing, as illustrated in Fig. 1(b). Moreover, from its mathematical expression, one sees that $(M_{DB})_{11}$ is the periodic function of w, with a period of $\tau = \frac{\pi}{k}$. For a fixed E, the tunneling probability $T[E; w]$ displays periodic variations with w, showing comb-like structures with the resonance peaks positioned at $L_n = a + w_n$, and the distance between any two neighboring peaks is $\Delta = w_n - w_{n-1} = \frac{\pi}{k}$ (Fig. 1(c)). The value of Δ is just half the de Broglie wavelength of the incident matter wave, indicating the key role of phase factor and quantum interference. Finally, the mathematical expression of w_n implies that there could be infinitely many resonance peaks in free space. The results may be readily extended to two- or three-dimensional

systems in that the interaction potentials along the direction of propagation are equivalently described by some effective double barriers.

B. TUNNELING ACROSS TYPICAL DOUBLE BARRIERS: RESONANT TUNNELING VERSUS SEQUENTIAL TUNNELING

For homo-structured rectangular double barriers, analytic expressions of w_n are available, which enable in-depth understanding of the physics of RT. The matrix element m_{11} describing the transmission across single rectangular barrier (barrier height V_0) may be expressed as follows (Appendix A):

$$m_{11} = 2\gamma e^{-ika}[i(k^2-\beta^2)\sinh(\beta a) + 2\beta k\cosh(\beta a)], \qquad (4)$$

where $k = \sqrt{2mE/\hbar^2}$, $\beta = \sqrt{2m(V_0-E)/\hbar^2}$, $\gamma = \frac{1}{4\beta k}$. Eq. (4) may be reduced to

$$m_{11} = 2\gamma e^{-ika} \times \sigma e^{i\alpha} = 2\gamma\sigma e^{i(\alpha-ka)}, \qquad (5)$$

where $\sigma = \sqrt{A^2+B^2}$, $A = (k^2-\beta^2)\sinh(\beta a)$, $B = 2\beta k\cosh(\beta a)$, and the angle $\alpha = \arctan(\frac{A}{B})$. Therefore, $m_{11}^2 = \frac{(A^2+B^2)}{4\beta^2k^2}e^{i2(\alpha-ka)}$. Using the theorem stated above, the angle $\theta = 2(\alpha-ka)$, and then $\theta + 2ka = 2\alpha$. The inter-barrier spacing is given by $w_n = \frac{n\pi}{k} - \frac{\pi+2\alpha}{2k}$. It follows that $2kw_n = (2n-1)\pi - 2\alpha$, and one arrives at the equality: $\tan(2kw_n) = \frac{\delta\tanh(\beta a)}{1-\frac{1}{4}\delta^2\tanh^2(\beta a)}$, where $\delta \equiv \left(\frac{\beta}{k} - \frac{k}{\beta}\right)$. Alternatively, this equality can be obtained by direct calculation of the squared norm of diagonal element $|(M_{DB})_{11}|^2$, a function of inter-barrier spacing w: The minimum of $|(M_{DB})_{11}|^2$ leads to RT (Appendix B). The equality for $\tan(2kw_n)$ is in line with Ref. [55], which was derived in a different way. In the special case when the incident energy is half the barrier height ($E = 0.5V_0$), $\beta = k$, the angle $\alpha = 0$, one obtains a simplified relation that $2kw_n = (2n-1)\pi$, and $w_n = \frac{(n-1/2)\pi}{k} = \left(n-\frac{1}{2}\right)\left(\frac{\lambda_d}{2}\right)$, with $\lambda_d = \frac{2\pi}{k}$, is the de Broglie wavelength. In another special case when $k \ll \beta$ and $\beta a \gg 1$, i.e., the incident energy is far below the barrier height, one has $\alpha \cong -\frac{\pi}{2} + \frac{k}{2\beta}$ and $kw_n \cong n\pi - \frac{k}{2\beta}$, $w_n \cong \frac{n\pi}{k} - \frac{1}{2\beta}$. In both situations, the value of w_n is independent of the barrier width.

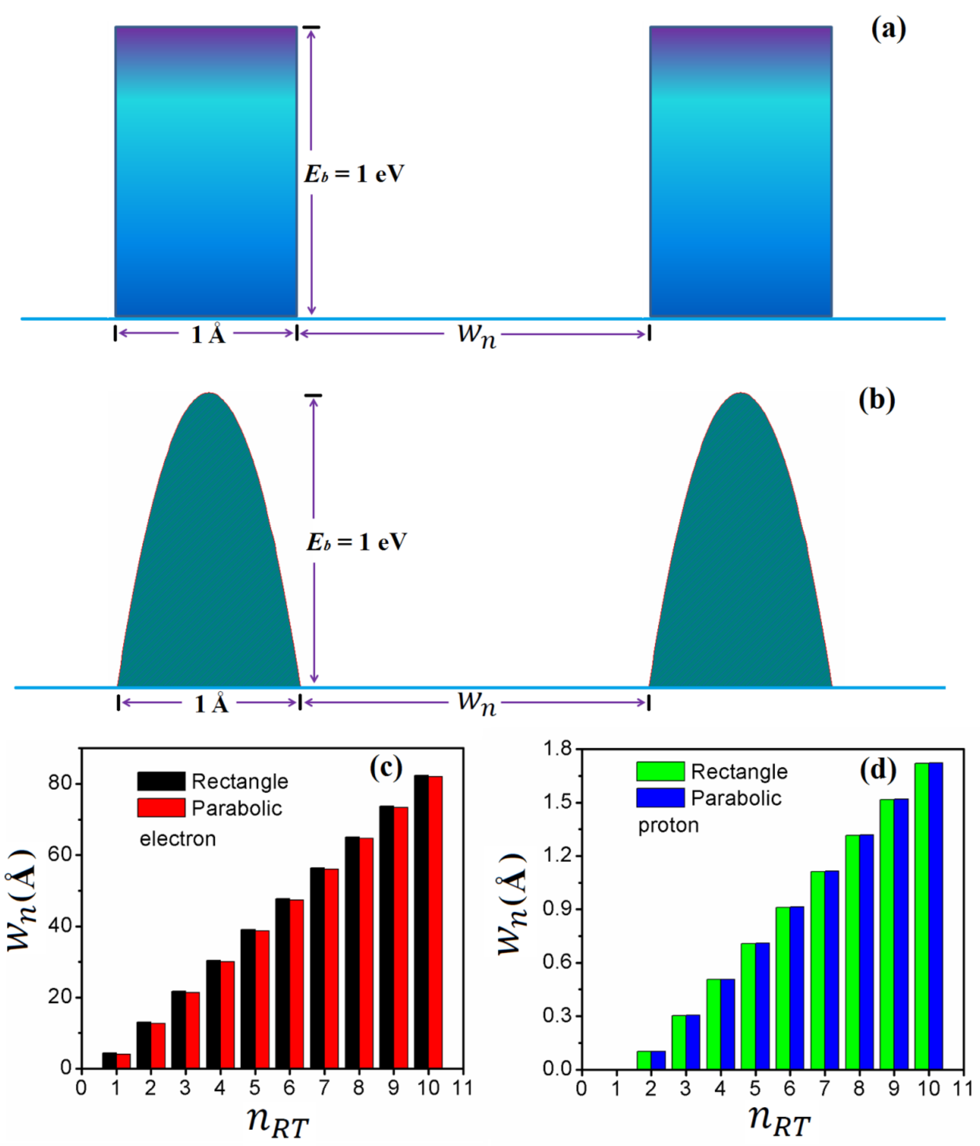

FIG. 2. Schematics of rectangular **(a)** and parabolic double barriers **(b)**. The inter-barrier spacing of RT (w_n), as a function of resonance number (n_{RT}), for electron **(c)** and proton **(d)** across the double barriers, at incident energy E = 0.5 eV.

The results are applicable to electrons and other massive quantum particles. However, demonstration of the quantum interference effects leading to RT would be much more challenging due to the large difference in particle masses and the corresponding de Broglie's wavelengths. Here, we perform systematic investigations on the RT characteristics of electrons and protons through two model systems: rectangular and parabolic double barriers (Fig. 2). Compared to the analytic

expressions for rectangular barriers, the transfer matrices for parabolic barriers are evaluated numerically [56, 57]. Figures 2(c)-(d) shows the calculated w_n for the RT of electrons and protons across the two types of double barriers, as a function of the resonance numbers (n_{RT}). For the same n_{RT}, the w_n of electrons is much larger than that of proton owing to smaller mass. The different geometries of the potential barriers (rectangular vs parabolic) are reflected by the slight differences of w_n. Despite the differences, the overall comparable magnitudes of the two sets of w_n indicate that rectangular double barriers may serve as approximations for qualitative description of some smoothly varying double barriers with regular geometries.

At fixed energy E, the tunneling probability varies periodically with inter-barrier spacing w. We have further studied such characteristics in case of electrons and protons tunneling through rectangular double barriers. Figures 3(a-b) show the transmission of electrons, at varying w for E = 0.03 eV and 0.5 eV. The effects of incident energy on the tunneling spectrum, $T(E; w)$, are clearly seen. Higher energy not only results in smaller period of oscillation ($\tau = \frac{\pi}{k}$), but also smaller peak-to-valley ratio. The resonance number can extend to very large integers, as long as the perturbation from the environment is negligible and the coherence of wave functions is maintained. To show the role of coherence, we have studied the energy-dependent tunneling probability $P(E)$ of electrons at a fixed inter-barrier spacing (w ~ 10μm). For resonant (coherent) tunneling, the quantity $P(E)$ (= $T(E; w)$) drops quickly with small deviations from the resonant energy level, E_{RT}. For sequential tunneling, in which the phase coherence is destroyed in a two-step tunneling process, the quantity $P(E)$ is simply product of the transmission coefficient across each single barriers: $P(E) = T_1(E) \times T_2(E) = T_1^2(E)$. Around the resonant energy E_{RT} (Figs. 3(c-d)), the probability of sequential tunneling (P_{ST}) changes smoothly with energy, and is significantly smaller than unity at low incident energies.

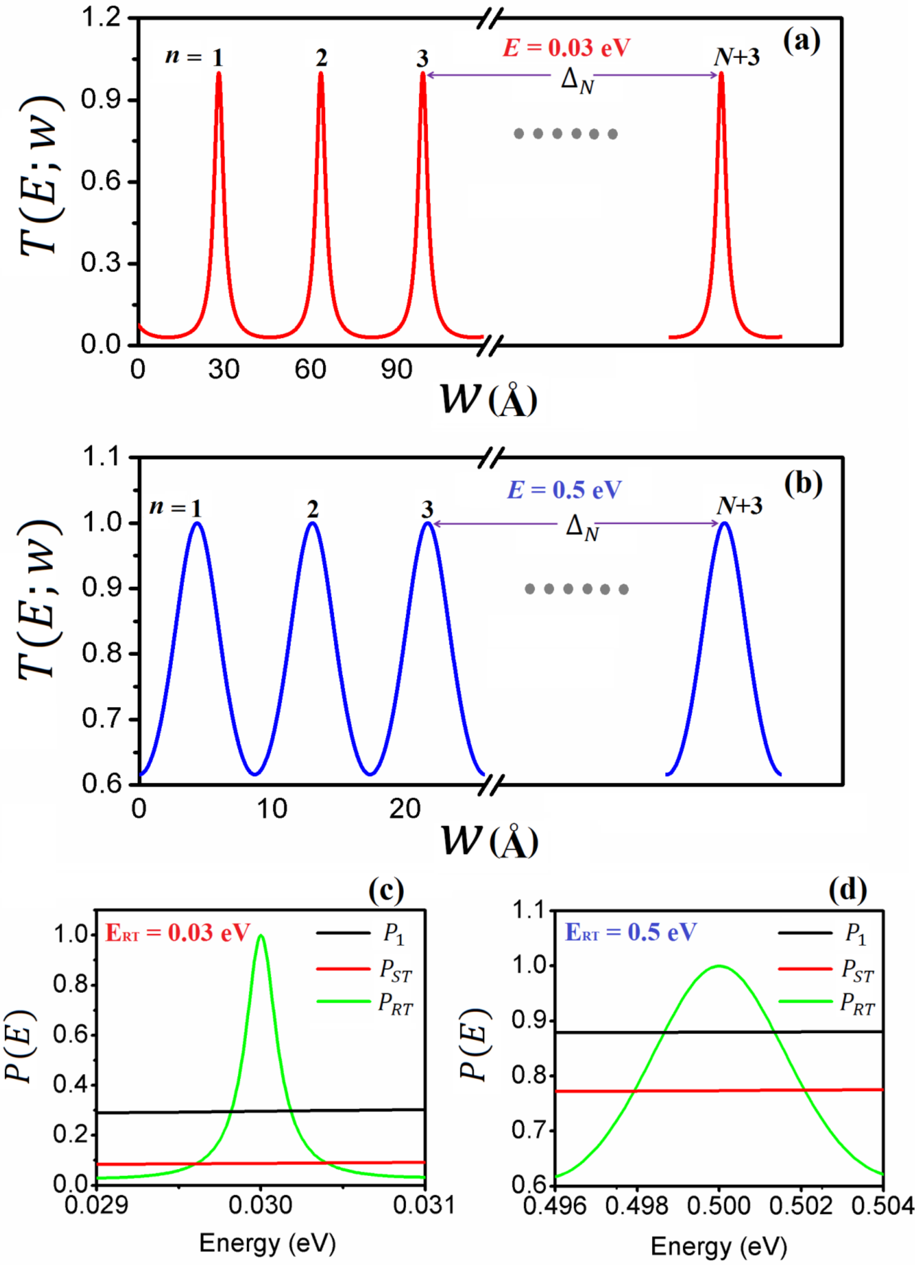

FIG. 3. Tunneling spectrum of electrons across the rectangular double-barrier shown in Fig. 2(a). RT at resonance level E_{RT} = 0.03 eV **(a)** and 0.5 eV **(b)**, as a function of inter-barrier spacing w. Panels **(c-d)**: Energy dependence of tunneling probability at fixed w, around E_{RT} = 0.03 eV (panel **c**, w = 100008.49 Å) and 0.5 eV (panel **d**, w = 100991.45 Å). The data lines labeled by P_1, P_{ST} and P_{RT} correspond to tunneling through single barrier, sequential and resonant tunneling through double-barrier, respectively.

For protons, more radical differences encounter. Shown in Fig. 4(a), is the

tunneling spectrum of protons at $E = E_b/2 = 0.5$ eV. The periodically repeated isolated lines imply much narrower resonant peaks with comparison to electrons. The enlarged structures of one resonant peak are shown in Fig. 4(b). Around the RT peaks, the squared norm of transfer matrix element may be expressed as follows (Appendix C):

$$|(M_{DB})_{11}|^2 \cong 1 + \sinh^2(2ka) \times (k\Delta w)^2 \equiv 1 + \Delta|M_{11}|^2_{\Delta w} \ , \tag{6}$$

where Δw is the deviation from the peak position w_n. When $\Delta|M_{11}|^2_{\Delta w} = 1$, $T(E; w) = 0.5$, and one has

$$|\Delta w| = \frac{1}{k\sinh(2ka)} \ . \tag{7}$$

It follows that the term $2|\Delta w|$ is the full width at half maximum (FWHM) of the resonant peaks. For the double-barrier (Fig. 2(a)) considered here, it turns out that $|\Delta w| \cong 4.235 \times 10^{-15}$ Å. When the deviation $\Delta w \sim 10^{-13}$ Å, the tunneling probability drops quickly to $T(E; w) \sim 10^{-3}$, in good agreement with the results presented in Fig. 4(b). In general, given that w is an approximate value to w_n, one can determine the significant digits of w by designating a deviation Δw when $|w - w_n| \leq \Delta w$ such that $T(E; w) \geq 1 - \delta P$, where δP $(0 < \delta P < 1)$ is the tolerance of decrease in tunneling probability at which significant tunneling (effective RT, events measurable in experiment) is maintained. Furthermore, using the proof of the theorem, we find that for arbitrarily homo-structured double barriers, the deviation Δw at the tolerance δP may be given by (Appendix D):

$$\Delta w = \frac{1}{2k}\sqrt{\frac{1}{R(1+R)} \times \frac{\delta P}{1-\delta P}} \ , \tag{8}$$

where $R = |m_{12}|^2$, k and δP are defined as above. Here we focus on the case that $\delta P = 0.5$, which yields the FWHM (= $2\Delta w$).

Such ultrahigh sensitivity on tunneling parameters is also found for the RT energies. Figure 4(c) compares the tunneling of protons across single and double barriers at a fixed inter-barrier spacing ($w \sim 20$ Å). Near resonance, $P(E)$ descends drastically from 1 to $\sim 10^{-11}$ by a tiny shift of $\varepsilon = 10^{-10}$ eV from E_{RT}. At the vicinity of E_{RT}, the dependence of $|(M_{DB})_{11}|^2$ with deviation ΔE is given by (Appendix C): $|(M_{DB})_{11}|^2 \cong 1 + \sinh^2(2ka) \times \left(\frac{kw}{2}\right)^2 \times \left(\frac{\Delta E}{E}\right)^2 \equiv 1 + \Delta|M_{11}|^2_{\Delta E}$. The FWHM at

energy scale is therefore obtained when $\Delta|M_{11}|^2_{\Delta E} = 1$, and $\left|\frac{\Delta E}{E}\right| = \frac{2}{(kw)\times\sinh(2ka)}$. In our case, $\left|\frac{\Delta E}{E}\right| \approx 4.235 \times 10^{-16}$. When the energy broadening $\Delta E = \varepsilon = 10^{-10}$eV, $\left|\frac{\Delta E}{E}\right| = 2 \times 10^{-10}$, $|(M_{DB})_{11}|^2 \cong 2.23 \times 10^{11}$, $P(E) = |(M_{DB})_{11}|^{-2} \approx 10^{-11.35}$, compares well with the numerical results. Without resonance, the probability of a two-step tunneling, i.e., sequential tunneling decreases by more than 25 orders of magnitude (Fig. 4(c)). The sharp contrast distinguishes RT from sequential tunneling. For the more generalized case, the allowed energy broadening ΔE may be calculated as follows (Appendix D):

$$\left|\frac{\Delta E}{E}\right| = \frac{1}{k(a+w)}\sqrt{\frac{1}{R(1+R)} \times \frac{\delta P}{1-\delta P}} \quad . \tag{9}$$

In the case of R >> 1 (large reflection), for instance, tunneling through large barriers or tunneling by massive particles, Eq. (9) is reduced to $\left|\frac{\Delta E}{E}\right| \cong \frac{1}{k(a+w)R}\sqrt{\frac{\delta P}{1-\delta P}}$. For a single barrier $V(x)$, the reflection and tunneling probabilities (Appendix D) are related to R by $|r|^2 = R|t|^2 \equiv RT_1(E)$, subjected to the condition $|r|^2 + |t|^2 = 1$. It is straightforward that $R = \frac{1}{|t|^2} - 1 \cong \frac{1}{|t|^2} = \frac{1}{T_1(E)}$ when R >> 1, where $T_1(E)$ is the tunneling probability across a single barrier.

As seen from Fig. 4(c), at the absence of phase coherence, the incident protons will be nearly completely reflected by a single barrier. On the contrary, when the inter-barrier spacing equals w_n and phase coherence is maintained, the protons penetrate the two barriers with a probability of unity. Such effects are schematically illustrated in Fig. 5. The key role of quantum interference is demonstrated. Experimental verifications may be carried out using atomically thin membranes, which have potential applications as proton sieve filters. Generally, the variation step (Δl) of the inter-barrier spacing w_n required by RT should be the order of magnitude of Δw studied above, and no less than the minimum length (i.e., $\Delta l \sim \Delta w \geq L_{min}$) such that effective RT can be reached by tuning the inter-barrier spacing. This is the topic of next subsection.

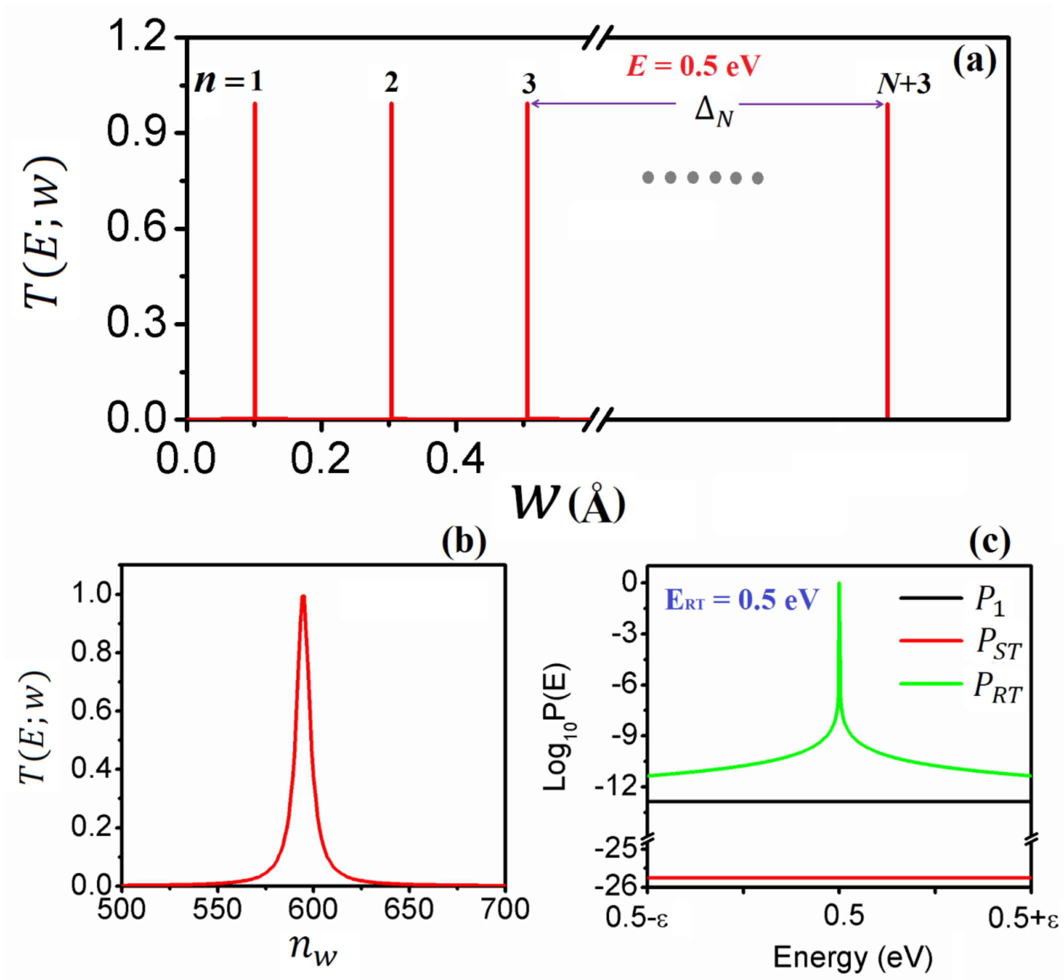

FIG. 4. Tunneling spectrum of protons across the rectangular double-barrier shown in Fig.2(a). Resonance at E_{RT} = 0.5 eV **(a)**. Panels **(b-c)**: Variations of tunneling probability with respect to small deviations from the RT parameters: **(b)** Inter-barrier spacing $w = (n_w - n_p)\times\Delta l + w_n$, $n_p = 596$ and $\Delta l = 10^{-15}$ Å; **(c)** Incident energy at the vicinity of E_{RT}, for w = 20.137016632763302 Å, and the energy deviation $\varepsilon = 10^{-10}$ eV. All digits of w are meaningful.

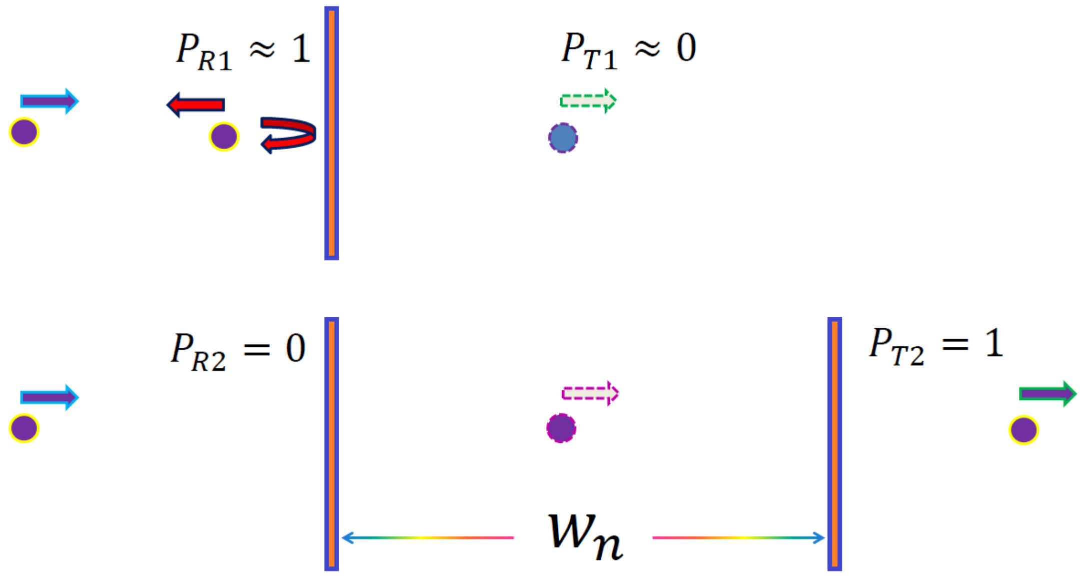

FIG. 5. Schematics of quantum tunneling of protons across single barrier (upper panel) and double barriers (lower panel) at the presence of RT. The probability of reflection is denoted by P_{Ri} and tunneling by P_{Ti}, i = 1, 2.

C. UPPER BOUNDS OF RT BARRIERS SET BY THE PLANCK LENGTH

The critical dependence of the tunneling probabilities on the barrier positions not only demonstrates the crucial role of phase factors, but also points to the possibility of ultrahigh accuracy measurements near resonance. As shown above, the deviation of $|\Delta w| \cong 4.235 \times 10^{-15}$ Å leads to a 50% drop of $P(E)$ of protons across a rectangular double-barrier. Such a deviation is several orders of magnitude below the smallest length scale sensed by LIGO [24, 25]. Even smaller $|\Delta w|$ is expected for heavier particles or larger barriers. As mentioned above, to have measurable RT within some tolerance δP, an upper bound of deviation from the exact peak position w_n is given by $\Delta w = |w - w_n|$. Suppose that the elementary variation step of distance is Δl; if the real space is a continuum ($L_{min} = 0$), then Δl can be arbitrarily small and in principle w_n can always be reached by a finite number of operations. In this case, the theorem stated above always holds. On the contrary, if a nonzero L_{min} exits ($L_{min} > 0$), to have significant RT, the variation step should satisfy $\Delta w \geq \Delta l \geq L_{min}$. Let $n = \lfloor \mathrm{Log}_{10}(\Delta w) \rfloor$, then it is a feasible choice to set $\Delta l = 10^n$, such that $w = N \times \Delta l$, with N being an integer and the modulo $w \bmod \Delta l = 0$. Consequently, the existence of a minimum

length leads to the inequality of $|\Delta w| \geq L_{min}$, and therefore puts some upper bounds for the particle mass, barrier height and barrier width, above which RT will cease. In the special case of tunneling across rectangular double barriers at $E = 0.5E_b$, realization of RT requires that $|\Delta w| = \frac{1}{k\sinh(2ka)} \geq L_{min}$, which may be rewritten as

$$\chi\sinh(\chi) \leq \frac{2a}{L_{min}}, \tag{10}$$

where $\chi = 2ka$, $k = \sqrt{\frac{2mE}{\hbar^2}} = \sqrt{\frac{mV_0}{\hbar^2}}$. The upper bound of the term mV_0 is therefore determined for a given barrier with a. Assuming that the minimum length is identical to the Planck length ($L_{min} = l_P$), the upper bounds of barrier height ($V_{\max}$) for electrons and protons are calculated and shown in Fig. 6. At a barrier width of a = 1 Å, 5 Å, and 10 Å, the $V_{\max}$ is ~ 5652.72 eV, 239.42 eV, and 61.32 eV for electrons, and is ~ 3.08 eV, 0.13 eV, and 0.03 eV for protons, respectively. It is seen that $V_{\max}$ of RT decreases fast with increasing barrier width. For instance, when the barrier width increases to a = 20 and 30 Å, the value of $V_{\max}$ is respectively ~ 15.70 and 7.08 eV for electrons, which may be feasible for experimental tests using metal-insulator-metal double barriers. Meanwhile, the same variation trend is found for electrons and protons, with the magnitude of $V_{\max}$ being scaled by a factor of $\eta = \frac{m_e}{m_p} \cong \frac{1}{1836}$, where m_e and m_p is respectively the mass of electron and proton. This is due to the conjugate relation that the particle mass times barrier height (mV_0) is a constant at fixed barrier width. For the general case of tunneling through arbitrary double barriers, the constraint imposed on the particle mass and barrier size due to a nonzero minimum length is given by (Appendix D):

$$\frac{\hbar}{2\sqrt{2mE}}\sqrt{\frac{1}{R(1+R)} \times \frac{\delta P}{1-\delta P}} \geq L_{min} \ , \tag{11}$$

where $R = |m_{12}|^2$, and δP $(0 < \delta P < 1)$ has the same meaning as above. Provided that $L_{min} = l_P$ and $\delta P = 0.5$, the inequality reduces to

$$\frac{\hbar}{2\sqrt{2mE}}\sqrt{\frac{1}{R(1+R)}} \geq l_P \ . \tag{12}$$

Since the parameter R is generally an increasing function of barrier size (Appendix D), the upper bounds on the barrier size of RT are therefore determined by the Planck length.

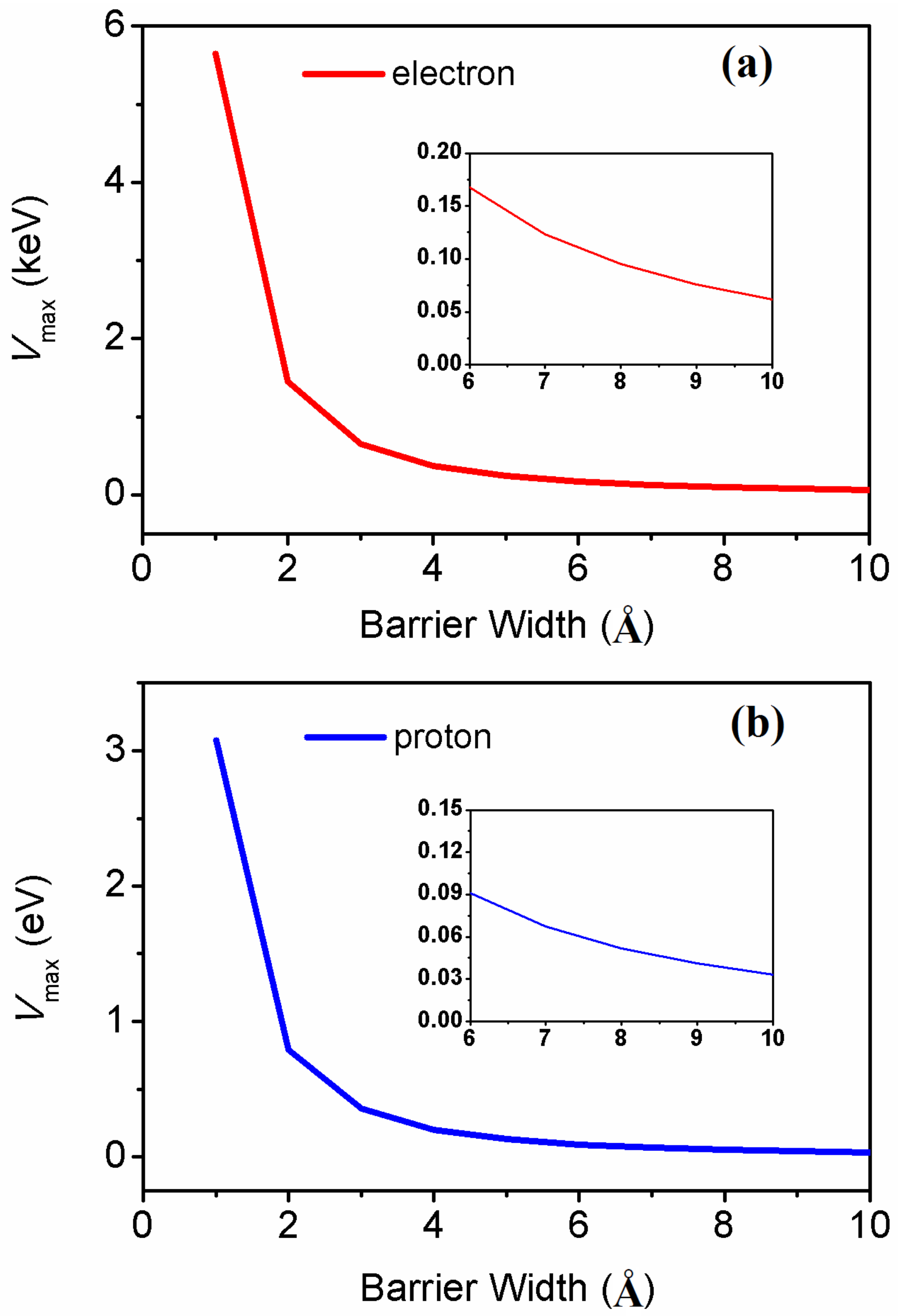

FIG. 6. Calculated upper bounds ($V_{\max}$) of the barrier height of rectangular double barriers set by the Planck length for electrons **(a)** and protons **(b)**, as a function of barrier width. The values of $V_{\max}$ at barrier width of 6-10 Å are highlighted in the insets.

D. FUNDAMENTAL LIMITS PUT BY THE UNCERTAINTY PRINCIPLE AND POSSIBLE SOLUTION

For a group of incident particles, given that the standard derivation of energy

distribution, σ_E, is approximately the term ΔE for $\Delta|M_{11}|^2_{\Delta E} = 1$ and $P(E) = 0.5$. The narrow window of energy dispersion implies that the momenta of the particles distribute dominantly within a narrow interval with a small standard derivation (Δp). As a consequence of the uncertainty principle, the standard derivation of position, Δx, is expected to be large. Table I lists the energy and momentum broadening, and estimated standard position derivations of protons when $P(E) = 0.5$ at $E = 0.5E_b$, for a number of rectangular double barriers with E_b = 1, 0.5, 0.2, and 0.1 eV, and w ~ 20 Å. It is clearly seen that the energy broadening increases significantly with decreasing barrier height, resulting in reduced standard derivations of position. For E_b = 1 eV, the requirement of ultrahigh monochromaticity of incident energies leads to a very small Δp and consequently a quite large Δx (~ 1.53×10^4 m), which is practically very challenging, if not impossible for experimental tests. Much smaller Δx (9.25×10^{-6} m) is found when E_b decreases to 0.1 eV.

It should be stressed here that, the constraint on the standard deviations of particle momentum and position imposed by the uncertainty principle *does not* exclude the possibility that a subgroup of particles with ultrahigh monochromaticity coexist with another subgroup of particles with large energy broadening. The reason is that, by definition, the standard derivation of some physical quantity of a single particle is the statistical average over a large number of events, which may be equivalently evaluated by the statistical results of a large number of identical particles within a small interval of time. Indeed, this is in line with the impossibility of measuring the quantum state of a single system [64]. Therefore, a possible recipe to the practical difficulty of position delocalization is to have much larger standard derivation of kinetic energy distribution than that required for the half drop of $P(E)$, i.e., $\sigma_E \gg \Delta E$, such that the standard derivation of momentum, σ_p, is much larger than the Δp corresponding to ΔE. Consider two microcanonical ensembles containing N weakly interacting identical bosons which follow the kinetic energy distributions $g_1(E)$ and $g_2(E)$ respectively: $N = \int_0^\infty g_1(E)dE = \int_0^\infty g_2(E)dE$. In addition, they have the same averaged kinetic energies: $\langle E\rangle = \int_0^\infty E g_1(E)dE = \int_0^\infty E g_2(E)dE$. The key

difference is the standard deviation of kinetic energies: $\sigma_{E2} \gg \sigma_{E1} \sim \Delta E$, i.e., the energy broadening of the first group of particles (distribution described by $g_1(E)$) is approximately the energy deviation for the half drop of $P(E)$, while is much smaller than that of the second group. The distribution function of the mixed $2N$-particle ensemble is $g(E) = g_1(E) + g_2(E)$, with the standard deviation $\sigma_E = \sqrt{\frac{\sigma_{E1}^2 + \sigma_{E2}^2}{2}} \approx \frac{\sigma_{E2}}{\sqrt{2}} \gg \Delta E$. Therefore, mixing of the two groups of identical particles has drastically increased the energy broadening and reduced the position uncertainty. Meanwhile, sufficient number of particles for resonant transmission is maintained. The modifications introduced by the procedure are illustrated in Fig. 7. For the general case of $P(E) = 1 - \delta P$, the energy broadening ΔE is given by Eq. (9) and can be similarly analyzed. In weakly interacting dilute atomic gases, two-body collisions dominate the interactions which simply exchange particle momenta and therefore keep the kinetic energy distributions unchanged.

In practice, the first group of particles may be prepared using the Bose-Einstein condensates [65-67], in which the momenta of all involved bosons are expected to have approximately the same value: Condensation in the momentum space. The second group of particles may be prepared at temperatures slightly above the critical temperature T_c of phase transition from normal states to the new quantum states like superconductivity, superfluidity, or Bose-Einstein condensation. As an example, the RT of some typical bosons (Cooper pair of superconducting Nb, ^{4}He, ^{7}Li, ^{23}Na, ^{87}Rb) across rectangular double barriers is studied and the related parameters are presented in Table II. The effects of energy broadening through mixing identical bosons of different ensembles are evidenced by the significantly reduced standard position deviations. Nevertheless, preparation of the first group of particles with ultrahigh monochromaticity remains challenging even with state-of-the-art technique. Another challenge to experimental tests may be the acceleration of the condensates as a whole to desired incident velocities while maintaining the states of condensation [68, 69].

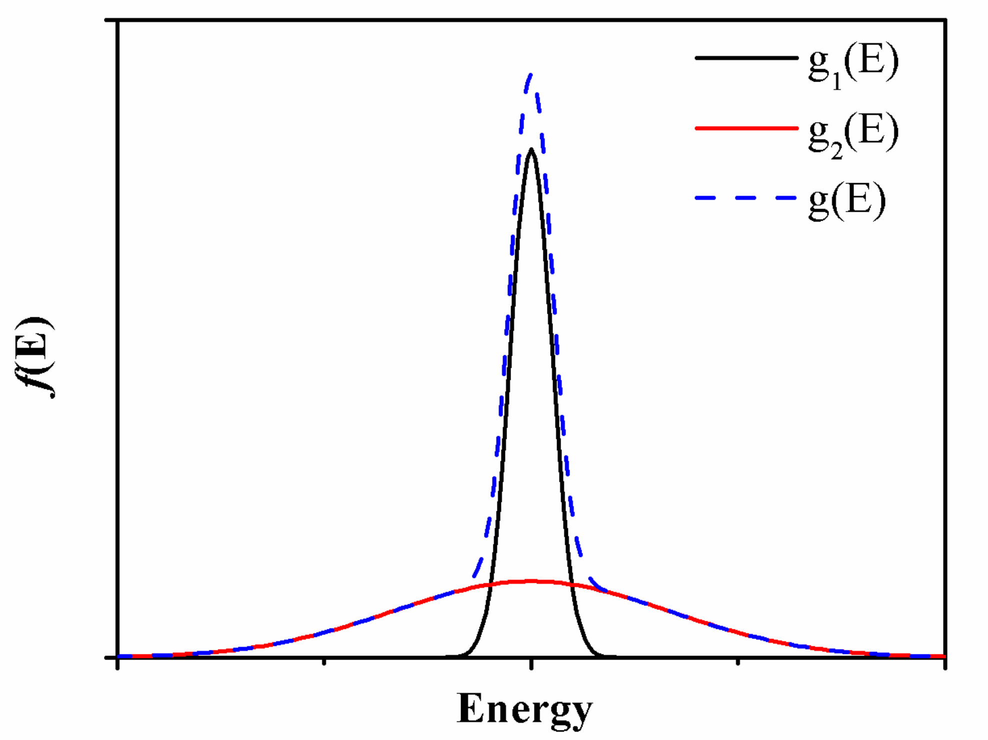

FIG. 7. Schematic diagram for the kinetic energy distribution (f(E)) of identical particles in microcanonical ensembles: Particle groups of high monochromaticity (f(E) = g_1(E)), low monochromaticity (f(E) = g_2(E)), and their superposition (f(E) = g(E) = g_1(E) + g_2(E)).

Table I. Parameters describing the RT of protons across rectangular double barriers at E = 0.5E_b. With the deviation of ΔE or $|\Delta w|$ from the parameters for resonance, the tunneling probability drops from 1 to 0.5. The corresponding momentum broadening Δp, and the minimum standard deviation of particle positions Δx_m are calculated using the relation $\Delta p \Delta x_m \geq \hbar/2$. In all cases the barrier width a = 1 Å.

E_b (eV)	w (Å)	ΔE (eV)	$\|\Delta w\|$ (Å)	Δp (kg.m/s)	Δx_m (m)
1	20.137016632763302	2.103×10^{-16}	4.235×10^{-15}	3.443×10^{-39}	15314.7
0.5	20.17790917547	1.320×10^{-12}	5.328×10^{-11}	3.056×10^{-35}	1.725
0.2	20.1380336	2.671×10^{-9}	2.690×10^{-7}	9.778×10^{-32}	5.392×10^{-4}
0.1	20.15963	1.101×10^{-7}	2.220×10^{-5}	5.700×10^{-30}	9.251×10^{-6}

Table II. Similar to Table I but for the RT of some typical bosons with incident energy E = $0.5E_b$. In all cases the barrier width a = 1 Å, and barrier height E_b = $0.01V_{max}$, with V_{max} being the upper bound set by the Planck length. The Cooper pairs of electrons are represented by $e^-\ldots e^-$. The energy broadening and resulted uncertainties of momenta and positions of mixed particle groups are displayed in the lower lines of the same columns. The broadening parameter of energy is chosen such that $\sigma_E \gtrsim k_B T_c$, with k_B the Boltzmann constant and T_c the phase transition temperatures.

Boson	E_b (eV)	w (Å)	$\|\Delta w\|$ (Å)	ΔE (eV)	Δp (kg.m/s)	Δx_m (m)
				σ_E (eV)	σ_p (kg.m/s)	σ_x (m)
$e^-\ldots e^-$ (in Nb)	28.26	6.3439	3.16×10^{-3}	1.41×10^{-12}	1.43×10^{-37}	368.15
				1×10^{-3}	1.02×10^{-28}	5.19×10^{-7}
^{4}He	7.69×10^{-3}	6.3439	3.16×10^{-3}	3.83×10^{-16}	1.43×10^{-37}	368.15
				5×10^{-4}	1.87×10^{-25}	2.82×10^{-10}
^{7}Li	4.39×10^{-3}	6.3439	3.16×10^{-3}	2.19×10^{-16}	1.43×10^{-37}	368.15
				1×10^{-10}	6.54×10^{-32}	8.07×10^{-4}
^{23}Na	1.34×10^{-3}	6.3439	3.16×10^{-3}	6.67×10^{-17}	1.43×10^{-37}	368.15
				1×10^{-10}	2.15×10^{-31}	2.45×10^{-4}
^{87}Rb	3.53×10^{-4}	6.3439	3.16×10^{-3}	1.76×10^{-17}	1.43×10^{-37}	368.15
				1×10^{-10}	8.13×10^{-31}	6.49×10^{-5}

III. CONCLUSIONS

To summarize, we have studied quantum tunneling across double barriers and arrived at a theorem which leads to several physical consequences. First of all, by tuning the inter-barrier spacing, it is possible that low-energy particles penetrate arbitrary finite-sized potential barriers completely via resonant tunneling (RT). This result points to the possibility of significant tunneling of massive quantum particles across large barriers at mild conditions. Secondly, it is possible to construct any desired quasi-bound energy levels within the quantum well formed by the two barriers via adjustment of the inter-barrier spacing. Thirdly, for the RT of quantum particles, it is possible to detect the tiny variations of energy levels and positions of the involved

potential barriers with unprecedented accuracies. Finally, the critical dependence on inter-barrier spacing (consequently the phase difference) demonstrates again the vital role of phase factor of wave function, which has manifested itself in some remarkable phenomenon such as the Aharonov-Bohm effect [70].

Demonstration of the above mentioned results involves two key factors: (i) Continuity of the real space and (ii) Energy monochromaticity of the incident particles. The first is determined by whether or not a nonzero minimum length (L_{min}) exists, and the second is affected by the uncertainty principle. Provided that L_{min} = 0, the distances in real space change continuously and RT can always be realized at given incident energies. On the contrary, the existence of a nonzero L_{min} will set constraints (upper bounds) for the particle mass, barrier height, and barrier width, beyond which no RT is expected. Meanwhile, to surmount the practical difficulty (position delocalization of incident particles) owing to the uncertainty principle, we suggest a plausible scheme in which the high- and low-monochromatic flows of identical particle groups are mixed. Potential applications of Bose-Einstein condensates in the scheme are discussed. This work reveals the deep connection between two seemingly different branches of quantum physics: quantum tunneling and quantum gravity, and opens a possible avenue for testing the existence of a minimum length.

ACKNOWLEDGEMENTS

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APPENDIX A: MATRIX ELEMENT m_{11} FOR TUNNELING ACROSS SINGLE RECTANGULAR BARRIER

Within the transfer matrix method, we derive the diagonal matrix element m_{11} that describes the transmission across a single rectangular barrier. For a quantum particle with incident energy E tunneling through a rectangular barrier with the height of V_0 and width a, the transfer matrix may be given by [56]:

$$M_1 = \frac{1}{2ik}\begin{pmatrix} (ik+\beta)e^{-(ik-\beta)a} & (ik-\beta)e^{-(ik+\beta)a} \\ (ik-\beta)e^{(ik+\beta)a} & (ik+\beta)e^{(ik-\beta)a} \end{pmatrix} \frac{1}{2\beta}\begin{pmatrix} \beta+ik & \beta-ik \\ \beta-ik & \beta+ik \end{pmatrix}$$
$$= \frac{\gamma}{i}\begin{pmatrix} (ik+\beta)^2 e^{-(ik-\beta)a} - (ik-\beta)^2 e^{-(ik+\beta)a} & (\beta^2+k^2)\left(e^{-(ik-\beta)a} - e^{-(ik+\beta)a}\right) \\ (\beta^2+k^2)\left(e^{(ik-\beta)a} - e^{(ik+\beta)a}\right) & (ik+\beta)^2 e^{(ik-\beta)a} - (ik-\beta)^2 e^{(ik+\beta)a} \end{pmatrix}$$

where $k = \sqrt{2mE/\hbar^2}$, $\beta = \sqrt{2m(V_0-E)/\hbar^2}$, $\gamma = \frac{1}{4\beta k}$.

The first diagonal term is: $m_{11} = \frac{\gamma}{i}[(ik+\beta)^2 e^{-(ik-\beta)a} - (ik-\beta)^2 e^{-(ik+\beta)a}] = -i\gamma[(ik+\beta)^2 e^{-(ik-\beta)a} - (ik-\beta)^2 e^{-(ik+\beta)a}]$, which can be reduced to

$m_{11} = -2i\gamma e^{-ika}[(\beta^2-k^2)\sinh(\beta a) + 2i\beta k\cosh(\beta a)]$, and finally one has

$$m_{11} = 2\gamma e^{-ika}[i(k^2-\beta^2)\sinh(\beta a) + 2\beta k\cosh(\beta a)] \qquad \text{(A1)}$$

APPENDIX B: DEDUCTION OF ALTERNATIVE RT CONDITION

In this appendix, we deduce the resonant tunneling (RT) condition for homo-structured rectangular double-barriers.

For a double-barrier (DB) consisting of single rectangular barriers with the height V_0 and barrier widths a and b, the diagonal element M_{11} of the transfer matrix M may be expressed as follows [57]:

$$|M_{11}|^2 =$$
$$1 + \frac{(\beta^2+k^2)^2}{4\beta^2k^2}[\sinh^2(\beta b) + \sinh^2(\beta a)] + 2\left[\frac{(\beta^2+k^2)^2}{4\beta^2k^2}\right]^2 \sinh^2(\beta b)\sinh^2(\beta a)$$
$$-\frac{1}{16k^4\beta^4}(\beta^2+k^2)^2\sinh(\beta b)\sinh(\beta a)\{\,[\,[(\beta^2+k^2)^2 - 8k^2\beta^2]\cosh\beta(a+b)$$
$$-(\beta^2+k^2)^2\cosh\beta(a-b)\,]\cdot\cos2[k(b+w)-ka]$$
$$-4k\beta(\beta^2-k^2)\sinh\beta(a+b)\cdot\sin2[k(b+w)-ka]\,\}. \qquad \text{(B1)}$$

where $k = \sqrt{2mE/\hbar^2}$, $\beta = \sqrt{2m(V_0-E)/\hbar^2}$, E is energy of the incident particle.

In the case of homo-structured DB, $a = b$, then

$$|M_{11}|^2 = 1 + \frac{(\beta^2+k^2)^2}{4\beta^2k^2} \times [2\sinh^2(\beta a)] + 2\left[\frac{(\beta^2+k^2)^2}{4\beta^2k^2}\right]^2 \sinh^4(\beta a) - \frac{1}{16k^4\beta^4}(\beta^2 + k^2)^2 \sinh^2(\beta a) \times \{[\,[(\beta^2+k^2)^2 - 8k^2\beta^2]\cosh(2\beta a) - (\beta^2+k^2)^2]\cos(2kw) - 4k\beta(\beta^2-k^2)\sinh(2\beta a)\cdot\sin(2kw)\}. \quad \text{(B2)}$$

For a given E, $|M_{11}|^2 = \frac{1}{T(E;\,w)}$ is the function of inter-barrier spacing w, the minimum of $|M_{11}|^2$ gives the maximum of transmission coefficient $T(E;\,w)$, i.e., resonant tunneling (RT). The condition of RT can be established by $\frac{\partial}{\partial w}|M_{11}|^2 = 0$. It follows that,

$$[\,[(\beta^2+k^2)^2 - 8k^2\beta^2]\cosh(2\beta a) - (\beta^2+k^2)^2] \times (-2k)\sin(2kw) - 4k\beta(\beta^2-k^2)\sinh(2\beta a) \times (2k)\cos(2kw) = 0,$$

and consequently

$$\tan(2kw) = \frac{4k\beta(\beta^2-k^2)\sinh(2\beta a)}{(\beta^2+k^2)^2 - [(\beta^2+k^2)^2 - 8k^2\beta^2]\cosh(2\beta a)} \quad \text{(B3)}$$

By dividing the term β^2k^2 in both numerator and denominator, Eq. (B3) changes to

$$\tan(2kw) = \frac{4\left(\frac{\beta}{k}-\frac{k}{\beta}\right)\sinh(2\beta a)}{\left(\frac{\beta}{k}+\frac{k}{\beta}\right)^2 - \left[\left(\frac{\beta}{k}+\frac{k}{\beta}\right)^2 - 8\right]\cosh(2\beta a)} \equiv \frac{4\delta\sinh(2\beta a)}{(\delta^2+4)-(\delta^2-4)\cosh(2\beta a)} = \frac{\delta\sinh(2\beta a)}{\left(1+\frac{1}{4}\delta^2\right)+\left(1-\frac{1}{4}\delta^2\right)\cosh(2\beta a)},$$

where $\delta \equiv \left(\frac{\beta}{k} - \frac{k}{\beta}\right)$.

Recalling that $\sinh(2\beta a) = 2\sinh(\beta a)\cosh(\beta a)$, $\cosh(2\beta a) = 2\cosh^2(\beta a) - 1$, one has

$$\tan(2kw) = \frac{2\delta\sinh(\beta a)\cosh(\beta a)}{\left(1+\frac{1}{4}\delta^2\right)+(1-\frac{1}{4}\delta^2)(2\cosh^2(\beta a)-1)} = \frac{2\delta\sinh(\beta a)\cosh(\beta a)}{\frac{1}{2}\delta^2+(1-\frac{1}{4}\delta^2)\times\cosh^2(\beta a)},$$

which can be reduced to

$$\tan(2kw) = \frac{\delta\tanh(\beta a)}{\left(1-\frac{1}{4}\delta^2\right)+\frac{1}{4}\frac{\delta^2}{\cosh^2(\beta a)}} = \frac{\delta\tanh(\beta a)}{\left(1-\frac{1}{4}\delta^2\right)+\frac{\delta^2}{4}\operatorname{sech}^2(\beta a)}. \quad \text{(B4)}$$

Using the equality $\operatorname{sech}^2(\beta a) = 1 - \tanh^2(\beta a)$, Eq.(B4) is finally reduced to

$$\tan(2kw) = \frac{\delta\tanh(\beta a)}{1-\frac{\delta^2}{4}\tanh^2(\beta a)} \quad \text{(B5)}$$

APPENDIX C: DEPENDENCE OF TUNNELING ON SMALL POSITION AND ENERGY CHANGES

In this appendix, we deduce the mathematical expressions describing the dependence of squared norm of diagonal transfer matrix element, $|(M_{DB})_{11}|^2$, with respect to slight deviations from the peak positions and incident energies at resonant tunneling (RT), for the special case when the incident energy is half the barrier height (V_0) of a homo-structured rectangular double-barrier (width of single barrier: a). The inverse of $|(M_{DB})_{11}|^2$ then describes the dependence of tunneling behavior on small position and energy changes.

In general, $|(M_{DB})_{11}|^2 \equiv f(E;w)$, is the function of incident energy E and inter-barrier spacing w. At the vicinity of RT, the function $f(E;w)$ can be expressed as functions of small deviations from RT parameters using the Taylor series, by considering the fact that $|(M_{DB})_{11}|^2 = 1$ and $\left(\frac{\partial f}{\partial w}\right) = 0$, $\left(\frac{\partial f}{\partial E}\right) = 0$ at the RT point.

I. For constant E, the dependence on deviation (Δw) from the RT positions (w_n) is

$$|(M_{DB})_{11}|^2 \equiv f(E;w) \cong 1 + \frac{1}{2}\left(\frac{\partial^2 f}{\partial w^2}\right) \times (\Delta w)^2 \equiv 1 + \Delta|M_{11}|^2_{\Delta w} \qquad \text{(C1)}$$

Using the expressions for rectangular double barriers (Appendix B), one has

$$\frac{\partial f}{\partial w} = -\left(\frac{1}{16\beta^4 k^4}\right)(\beta^2 + k^2)^2 \sinh^2(\beta a)[\mathrm{g}(\beta,k) \times (-2k) \times \sin(2kw) + \mathrm{h}(\beta,k) \times (2k) \times \cos(2kw)], \qquad \text{(C2)}$$

$$\frac{\partial^2 f}{\partial w^2} = \frac{(\beta^2+k^2)^2 \sinh^2(\beta a)}{4\beta^4 k^2}[\mathrm{g}(\beta,k)\cos(2kw) + \mathrm{h}(\beta,k)\sin(2kw)], \qquad \text{(C3)}$$

where $k = \sqrt{2mE/\hbar^2}$, $\beta = \sqrt{2m(V_0 - E)/\hbar^2}$,

$\mathrm{g}(\beta,k) \equiv [(\beta^2 + k^2)^2 - 8\beta^2 k^2]\cosh(2\beta a) - (\beta^2 + k^2)^2$, $\mathrm{h}(\beta,k) = -4\beta k(\beta^2 - k^2)\sinh(2\beta a)$.

The condition $\left(\frac{\partial f}{\partial w}\right) = 0$ gives that $\mathrm{g}(\beta,k)\sin(2kw) = \mathrm{h}(\beta,k)\cos(2kw)$, and then

$$\tan(2kw) = \frac{\mathrm{h}(\beta,k)}{\mathrm{g}(\beta,k)} \qquad \text{(C4)}$$

Using Eq. (C4), $\frac{\partial^2 f}{\partial w^2} = \frac{(\beta^2+k^2)^2 \sinh^2(\beta a)}{4\beta^4 k^2}\mathrm{h}(\beta,k)\sin(2kw)[\cot^2(2kw) + 1]$, and then

$$\frac{\partial^2 f}{\partial w^2} = \frac{(\beta^2+k^2)^2 \sinh^2(\beta a)}{4\beta^4 k^2} \times \frac{\mathrm{h}(\beta,k)}{\sin(2kw)} \qquad \text{(C5)}$$

For rectangular double barriers, we have the general relation $2kw = (2n-1)\pi - 2\alpha$, and $\alpha = \arctan[\frac{(k^2-\beta^2)}{2\beta k}\tanh(\beta a)]$.

Consequently,

$$\sin(2kw) = \sin(2\alpha) = \frac{2\tan\alpha}{1+\tan^2\alpha} = \frac{2\frac{(k^2-\beta^2)}{2\beta k}\tanh(\beta a)}{1+\tan^2\alpha}.$$

Finally,

$$\frac{\partial^2 f}{\partial w^2} = \frac{\left(\beta^2+k^2\right)^2 \sinh^2(\beta a)}{4\beta^4 k^2} \frac{\mathrm{h}(\beta,k)}{\sin(2kw)} = \frac{\left(\beta^2+k^2\right)^2 \sinh^2(2\beta a)}{2\beta^2}(1+\tan^2\alpha) \qquad \text{(C6)}$$

It is clear that $\frac{\partial^2 f}{\partial w^2} > 0$ holds for all allowed incident energies E, *which proves that the term* $|(M_{DB})_{11}|^2$ *arrives at its minimum and its reciprocal gives the maximum of transmission probability, i.e.*, 1.

When the incident energy is half the barrier height, $\beta = k$, we have $\alpha = 0$, and $\frac{\partial^2 f}{\partial w^2} = 2k^2\sinh^2(2ka)$, and therefore

$$|(M_{DB})_{11}|^2 \cong 1 + \frac{1}{2}\left(\frac{\partial^2 f}{\partial w^2}\right) \times (\Delta w)^2 = 1 + \sinh^2(2ka) \times (k\Delta w)^2 \qquad \text{(C7)}$$

II. For constant w, the dependence on deviation (ΔE) from the RT energies (E_{RT}) is

$$|(M_{DB})_{11}|^2 \equiv f(E;w) \cong 1 + \frac{1}{2}\left(\frac{\partial^2 f}{\partial E^2}\right) \times (\Delta E)^2 \equiv 1 + \Delta|M_{11}|^2_{\Delta E} \qquad \text{(C8)}$$

Compared to $\left(\frac{\partial^2 f}{\partial w^2}\right)$, computation of $\left(\frac{\partial^2 f}{\partial E^2}\right)$ is much more complicated. Alternatively, we directly consider the dependence of $|(M_{DB})_{11}|^2$ with energy deviation (ΔE) to the second order. For the special situation $\beta = k$, the mathematical expression of $|(M_{DB})_{11}|^2$ is reduced to

$$|(M_{DB})_{11}|^2 = 1 + 2\sinh^2(ka) + 2\sinh^4(ka) + \sinh^2(ka)[\cosh(2ka) + 1] \times \cos(2kw) \qquad \text{(C9)}$$

Recalling that $2kw = (2n-1)\pi$ for $\beta = k$, the term $\cos(2kw)$ may be expressed by Taylor series around the RT point with respect to Δk *to the second order*:

$$\cos(2kw) \cong -1 + \frac{1}{2}(2w)^2(\Delta k)^2 = -1 + 2(w\Delta k)^2 \qquad \text{(C10)}$$

Substitution of $\cos(2kw)$ with Eq. (C10) leads to the following

$$|(M_{DB})_{11}|^2 \cong 1 + \sinh^2(2ka)(w\Delta k)^2 \qquad \text{(C11)}$$

Using $k = \sqrt{2mE/\hbar^2}$, and then $\Delta k = \frac{\Delta E}{2\sqrt{E}}\sqrt{2m/\hbar^2} = \frac{k}{2} \times \frac{\Delta E}{E}$, one finally arrives at

$$|(M_{DB})_{11}|^2 \cong 1 + \sinh^2(2ka)(\frac{kw}{2})^2(\frac{\Delta E}{E})^2 \qquad \text{(C12)}$$

APPENDIX D: GENERALIZED CONSTRAINTS ON BARRIER SIZE DUE TO A MINIMUM LENGTH

In this appendix, we deduce the constraint on the barrier size (barrier height, barrier width) for effective resonant tunneling (RT) at the presence of a nonzero minimum length ($L_{\min}$). As defined above, effective RT implies that giving the deviation Δw when $|w - w_n| \leq \Delta w$, the inequality $T(E; w) \geq 1 - \delta P$ holds, where δP $(0 < \delta P < 1)$ is the tolerance of decrease in tunneling probability at which significant tunneling is measurable. Based on the proof of the theorem, we have

$$T_{DB}(E; w) = \frac{1}{|(M_{DB})_{11}|^2} = \frac{1}{\left|e^{i\theta}[1+R(e^{-i(\phi+\theta)}+1)]\right|^2} = \frac{1}{\left|1+R(e^{-i(\phi+\theta)}+1)\right|^2} . \qquad \text{(D1)}$$

Then Δw is determined by the equality as follows

$$\frac{1}{\left|1+R(e^{-i(\phi+\theta)}+1)\right|^2} = 1 - \delta P. \qquad \text{(D2)}$$

Equivalently,

$$\left|1 + R(e^{-i(\phi+\theta)} + 1)\right|^2 = \frac{1}{1-\delta P} . \qquad \text{(D3)}$$

It follows that

$$|1 + R + R(\cos(\phi + \theta) - i\sin(\phi + \theta))|^2 = \frac{1}{1-\delta P} , \qquad \text{(D4)}$$

$$(1 + R + R\cos(\phi + \theta))^2 + R^2\sin^2(\phi + \theta) = \frac{1}{1-\delta P} , \qquad \text{(D5)}$$

$$(1 + R)^2 + R^2 + 2(1 + R)R\cos(\phi + \theta) = \frac{1}{1-\delta P} , \qquad \text{(D6)}$$

and then reduces to

$$2R(1 + R)\,[1 + \cos(\phi + \theta)] = \frac{\delta P}{1-\delta P} . \qquad \text{(D7)}$$

Consequently, we arrive at

$$\cos(\phi + \theta) = -1 + \frac{1}{2R(1+R)} \times \frac{\delta P}{1-\delta P} . \qquad \text{(D8)}$$

For a homo-structured double-barrier system, the two parameters $\theta = \arg({m_{11}}^2)$, and $R = |m_{12}|^2$ are solely determined by a single barrier $V(x)$. The tunable parameter is $\phi = 2k(a + w)$, via variation of the inter-barrier spacing w by a small

magnitude of Δw. At the vicinity of RT, $|\Delta w| << w_n$. Around $\phi + \theta = (2n-1)\pi$, i.e., the RT points, expansion of $\cos(\phi+\theta)$ using Taylor series to the 2nd order, we have

$$\cos(\phi+\theta) \cong -1 + \frac{1}{2}(\Delta\phi)^2 \;, \tag{D9}$$

where $\Delta\phi = \pm 2k\Delta w$. Comparison of Eq. (D8) and (D9) gives that

$$2k\Delta w = \sqrt{\frac{1}{R(1+R)} \times \frac{\delta P}{1-\delta P}} \;. \tag{D10}$$

Finally, we get

$$\Delta w = \frac{1}{2k}\sqrt{\frac{1}{R(1+R)} \times \frac{\delta P}{1-\delta P}} \;. \tag{D11}$$

To achieve effective RT, the existence of $L_{\min}$ requires that

$$\Delta w = \frac{1}{2k}\sqrt{\frac{1}{R(1+R)} \times \frac{\delta P}{1-\delta P}} \geq L_{min} \tag{D12}$$

For a single barrier $V(x)$, the reflection coefficient is given by [59-62]

$$|r|^2 = \frac{|m_{12}|^2}{|m_{11}|^2} = RT_1(E) = R|t|^2, \tag{D13}$$

where $R = |m_{12}|^2$, and $T_1(E) = |t|^2$ is the transmission coefficient across $V(x)$ at energy E. Conservation of probability current gives that $|r|^2 + |t|^2 = 1$. Qualitatively, $|r|^2$ increases with barrier width a and barrier height E_b, which indicates that R *is the increasing function of barrier size* parameters a and E_b: $R = R\,(a, E_b)$. Larger barrier size results in larger value of R. Substitution of k with $\frac{\sqrt{2mE}}{\hbar}$, the inequality (D12) is therefore

$$\frac{\hbar}{2\sqrt{2mE}}\sqrt{\frac{1}{R(1+R)} \times \frac{\delta P}{1-\delta P}} \geq L_{min} \tag{D14}$$

This is the constraint imposed on the particle mass, barrier height, and barrier width due to the minimum length.

In the case $\delta P = 0.5$, FWHM (= $2\Delta w$) is obtained. Given that $L_{min} = l_P$, we have

$$\frac{\hbar}{2\sqrt{2mE}}\sqrt{\frac{1}{R(1+R)}} \geq l_P \tag{D15}$$

For a fixed particle mass m and incident energy E, the inequality (D15) sets upper bounds on R and consequently the upper bounds for barrier size of $V(x)$: the barrier

width a and barrier height E_b.

Furthermore, we can derive the constraint on the broadening of incident energy by using Eq. (D9). In this case, w (= w_n) and a are fixed, $\Delta\phi = 2\Delta k(a + w)$. Using $k = \frac{\sqrt{2mE}}{\hbar}$, we have $\Delta k = \frac{k}{2} \times \frac{\Delta E}{E}$, then $2\Delta k = k \times \frac{\Delta E}{E}$, and $\Delta\phi = k(a + w) \times \frac{\Delta E}{E}$. It follows that

$$(\Delta\phi)^2 = \frac{1}{R(1+R)} \times \frac{\delta P}{1-\delta P} \quad . \tag{D16}$$

Then

$$\Delta\phi = k(a + w) \times \left|\frac{\Delta E}{E}\right| = \sqrt{\frac{1}{R(1+R)} \times \frac{\delta P}{1-\delta P}} \quad . \tag{D17}$$

Finally we have

$$\left|\frac{\Delta E}{E}\right| = \frac{1}{k(a+w)} \sqrt{\frac{1}{R(1+R)} \times \frac{\delta P}{1-\delta P}} \tag{D18}$$

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